

Bounds for the free energy of directed polymers in random environment in dimension 1+1

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Abstract

We give sharp estimates for the free energy of directed polymers in random environment in dimension 1+1. These estimates were known for a gaussian environment, we extend them to the case where the law of the environment is infinitely divisible.

Key words. directed polymers in random environment, free energy, infinite divisibility, FKG inequality

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1 Introduction

We refer to [5] for a review of directed polymers in random environment. Let $S = (S_n)_{n \in \mathbb{N}}$ be the simple random walk on \mathbb{Z}^d starting at 0, defined on the probability space $(\Sigma, \mathcal{E}, \mathbf{P})$. Let $\eta = (\eta(n, x))_{(n, x) \in \mathbb{N} \times \mathbb{Z}^d}$ be a sequence of real-valued i.i.d. random variables defined on another probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The expectation of a function f with respect to the probability measure \mathbf{P} (respectively \mathbb{P}) will be denoted by $\mathbf{E}f = \int_{\Sigma} f d\mathbf{P}$ (respectively $\mathbb{E}f = \int_{\Omega} f d\mathbb{P}$). The path $(i, S_i)_{1 \leq i \leq n}$ represents the directed polymer of size n in dimension $1 + d$, and η is the random environment. For β strictly positive, featuring the inverse of the temperature, we define the random polymer probability measure $\mathbf{P}_{n, \beta}$ on the path space (Σ, \mathcal{E}) by its density with respect to \mathbf{P}

$$\frac{d\mathbf{P}_{n, \beta}}{d\mathbf{P}}(S) = \frac{1}{Z_n(\beta)} \exp(\beta H_n(S)), \quad (1.1)$$

where

$$H_n(S) = \sum_{j=1}^n \eta(j, S_j), \text{ and } Z_n(\beta) = \mathbf{E} \exp(\beta H_n(S)). \quad (1.2)$$

For a given realisation of the environment η , the measure $\mathbf{P}_{n, \beta}$ gives weight to polymers $(i, S_i)_{1 \leq i \leq n}$ with low energy $-\beta H_n(S)$ (configurations of lowest energy are the most probable). For simplicity we write $\mathbb{E}f(\eta) = \mathbb{E}f(\eta(0, 0))$ for any f such that $f \circ \eta(0, 0)$ is integrable. Let $\lambda(\beta) = \ln \mathbb{E} e^{\beta \eta}$ be the logarithmic moment generating function of η . We suppose that there exists $B > 0$ such that

$$\mathbb{E} e^{B|\eta|} < \infty \text{ for } 0 \leq \beta \leq B. \quad (1.3)$$

It is well known that the sequence $\mathbb{E} \ln Z_n(\beta)$ is superadditive, hence the limit

$$p(\beta) = \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E} \ln(Z_n(\beta)) = \sup_n \frac{1}{n} \mathbb{E} \ln(Z_n(\beta)) \in (-\infty, \lambda(\beta)] \quad (1.4)$$

exists. It is called the free energy of the polymer. We proved in [12], [14] that if $\mathbb{E} e^{\beta|\eta|} < \infty$ for a fixed $\beta > 0$, then there exists $K > 0$ such that for all $n \geq 1$,

$$\mathbb{P}(\pm \frac{1}{n} (\ln Z_n(\beta) - \mathbb{E} \ln Z_n(\beta)) > x) \leq \begin{cases} \exp(-\frac{nx^2}{4K}) & \text{if } x \in (0, 2K], \\ \exp(-n(x - K)) & \text{if } x \in (2K, \infty). \end{cases}$$

This concentration property implies that under our hypothesis, $\frac{1}{n} \ln Z_n(\beta)$ converges \mathbb{P} a.s. towards $p(\beta)$ for every $\beta \in (0, B)$. This was first proved by Carmona and Hu ([3], Proposition 1.4) for a gaussian environment, and by Comets, Shiga and Yoshida ([4], Proposition 2.5) for a general environment, but under the stronger condition that $\mathbb{E} e^{\beta|\eta|} < \infty$ for every $\beta > 0$.

We consider the normalized partition function

$$W_n(\beta) = \frac{Z_n(\beta)}{\mathbb{E} Z_n(\beta)} = Z_n(\beta) e^{-n\lambda(\beta)},$$

and the normalized free energy

$$p_-(\beta) = \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E} \ln(W_n(\beta)) = p(\beta) - \lambda(\beta) \in (-\infty, 0].$$

Bolthausen [1] noticed that (W_n) is a positive martingale, hence converges \mathbb{P} a.s. towards a variable W_∞ , and that moreover $\mathbb{P}(W_\infty = 0)$ is 0 or 1. When there is no random environment, the normalized partition function W_n is constantly equal to one. Accordingly we say that weak disorder holds when $\mathbb{P}(W_\infty = 0) = 0$, strong disorder holds when $\mathbb{P}(W_\infty = 0) = 1$. It is immediate that if weak disorder holds, then the normalized free energy $p_-(\beta)$ equals zero. Comets and Yoshida proved monotonicity in β concerning both the dichotomy between weak and strong disorder, and the normalized free energy.

Theorem ([7], Theorem 3.2)

1. *There exists a critical value $\beta_0 = \beta_c(d) \in [0, \infty]$ with $\beta_0 = 0$ for $d = 1, 2$, $0 < \beta_0 \leq \infty$ for $d \geq 3$, such that $\mathbb{P}(W_\infty = 0) = 0$ if $\beta \in \{0\} \cup (0, \beta_0)$, $\mathbb{P}(W_\infty = 0) = 1$ if $\beta > \beta_0$.*
2. *The normalized free energy $p_-(\beta)$ is non-increasing in $\beta \in [0, \infty)$. In particular there exists β_c with $\beta_0 \leq \beta_c \leq \infty$ such that $p_-(\beta) = 0$ if $0 \leq \beta \leq \beta_c$, $p_-(\beta) < 0$ if $\beta > \beta_c$.*

It is widely believed that the two critical values β_0 and β_c should be equal, but it is still an open question in dimension d greater or equal to three. The equality $\beta_0 = \beta_c = 0$ is true in dimension one and two. This was proved by Comets and Vargas ([6]) for the dimension one, and by Lacoïn ([10]) for the dimension two. Moreover in this paper Lacoïn improves the result of Comets and Vargas by giving sharp estimates of the normalized free energy at high temperature (that is for small β). He proves the following theorem.

Theorem ([10], Theorem 1.4 and Theorem 1.5)

1. When $d = 1$ and (1.3) holds, there exist constants c and $B_0 < B$ such that for every β in $[0, B_0)$,

$$-\frac{1}{c}\beta^4[1 + (\ln \beta)^2] \leq p_-(\beta) \leq -c\beta^4. \quad (1.5)$$

2. When $d = 1$ and the environment is gaussian, then there exists a constant c such that for every $\beta \leq 1$,

$$-\frac{1}{c}\beta^4 \leq p_-(\beta) \leq -c\beta^4. \quad (1.6)$$

Our aim is to get rid of the logarithmic factor of the lower bound in the case of a general environment. To prove the lower bound in (1.5) Lacoïn combines the second moment method and a directed percolation argument, whereas in (1.6) he uses a specific gaussian approach similar to what is done in [13]. More precisely his proof relies on two inequalities, both obtained using gaussian integration by parts. We are able to generalize these inequalities in the case of an infinitely divisible environment. The first inequality is still obtained by an integration by parts formula, valid for infinitely divisible distributions, that we learned in [2]. To prove the second inequality we use the FKG inequality in the manner of [7].

THEOREM 1.1 *When $d = 1$ and η has an infinitely divisible distribution satisfying (1.3), there exist constants $C > 0$ and $B_0 < B$ such that for every β in $(0, B_0)$,*

$$-C\beta^4 \leq p_-(\beta). \quad (1.7)$$

The paper is organized as follows. In the first part we explain the proof of Theorem 1.1, reducing it to the proof of the two inequalities mentioned above. In the second part we prove the first inequality, using the integration by parts formula for infinitely divisible distributions (Lemma 2.1). In the third part we prove the second inequality by using the FKG inequality (Lemma 2.2).

2 Proof of Theorem 1.1

We can assume without loss of generality that the variance σ^2 of η is strictly less than one. If it is not the case, we consider $\tilde{\eta} = \frac{\eta}{2\sigma}$ whose variance is $\frac{1}{4}$ and we use that $p_-(\beta) = \tilde{p}_-(2\sigma\beta)$. We define

$$p_n(\beta) = \frac{1}{n} \mathbb{E} \ln W_n(\beta) = \frac{1}{n} \mathbb{E} \ln Z_n(\beta) - \lambda(\beta) \quad (2.8)$$

and we recall that

$$p_-(\beta) = \sup_{n \geq 1} p_n(\beta).$$

We use the same strategy as Lacoïn in [10] and show that there exist $B_1 > 0$ and $c > 0$ such that for every $n \geq 1$ and every $\beta \in [0, B_1)$,

$$p_n(\beta) \geq (1 - e^c) \frac{1}{2n} \ln E^{\otimes 2} e^{2\beta^2 L_n(S^1, S^2)}, \quad (2.9)$$

where S^1, S^2 are two independent copies of the simple random walk, $E^{\otimes 2}$ is the expectation on the product space $(\Sigma^2, \mathcal{E}^{\otimes 2})$, and

$$L_n(S^1, S^2) = \sum_{i=1}^n \mathbf{1}_{S_i^1 = S_i^2}$$

is the number of intersections of the two directed paths $(i, S_i^1)_{1 \leq i \leq n}$ and $(i, S_i^2)_{1 \leq i \leq n}$. Letting n tend to infinity in (2.9) we find that

$$p_-(\beta) \geq \frac{1 - e^c}{2} F(2\beta^2),$$

where

$$F(t) = \lim_{n \rightarrow \infty} \frac{1}{n} \ln E^{\otimes 2} e^{t L_n(S^1, S^2)}$$

is the free energy of the homegeneous pinning model. We refer to [8] for a survey of this model, where it is proved that

$$F(t) \sim \frac{t^2}{2} \text{ when } t \rightarrow 0^+.$$

This implies that for any $C > e^c - 1$, there exists $B_0 < B$ such that for all $0 \leq \beta < B_0$,

$$p_-(\beta) \geq -C\beta^4,$$

which proves the lower bound result (1.7). To prove (2.9) we proceed as in [10] by interpolation between the two functions

$$p_n(\beta) = \frac{1}{n} \mathbb{E} \ln E e^{\beta H_n(S) - n\lambda(\beta)} = \frac{1}{2n} \mathbb{E} \ln E^{\otimes 2} e^{\beta H_n(S^1, S^2) - 2n\lambda(\beta)},$$

where

$$H_n(S^1, S^2) = H_n(S^1) + H_n(S^2),$$

and

$$F_n(\beta) = \frac{1}{2n} \ln E^{\otimes 2} e^{2\beta^2 L_n(S^1, S^2)} = \frac{1}{2n} \mathbb{E} \ln E^{\otimes 2} e^{2\beta^2 L_n(S^1, S^2)}.$$

Let β be fixed in $]0, B[$. We define, for $t \in [0, 1]$ and $u \geq 0$,

$$\varphi_n(t, u) = \frac{1}{2n} \mathbb{E} \ln E^{\otimes 2} e^{\sqrt{t}\beta H_n(S^1, S^2) - 2n\lambda(\sqrt{t}\beta) + u\beta^2 L_n(S^1, S^2)},$$

$$\phi_n(t) = \varphi_n(t, 0) = \frac{1}{n} \mathbb{E} \ln E e^{\sqrt{t}\beta H_n(S) - n\lambda(\sqrt{t}\beta)} = p_n(\sqrt{t}\beta),$$

so that

$$\phi_n(1) = \varphi_n(1, 0) = p_n(\beta), \quad \varphi_n(0, 2) = F_n(\beta).$$

The inequality (2.9) amounts to

$$\phi_n(1) \geq (1 - e^c) \varphi_n(0, 2). \tag{2.10}$$

We shall prove that

$$\forall t \in [0, 1], \quad \phi_n'(t) \geq c(\phi_n(t) - \varphi_n(0, 2)), \tag{2.11}$$

which by Gronwall's lemma implies (2.10). We write

$$\phi_n(t) - \varphi_n(0, 2) = \varphi_n(t, 0) - \varphi_n(t, 2-t) + \varphi_n(t, 2-t) - \varphi_n(0, 2) \quad (2.12)$$

and we consider separately the two differences of the right member.

The function $u \mapsto \varphi_n(t, u)$ is convex non decreasing, hence for every $t \in [0, 1]$,

$$\varphi_n(t, 2-t) - \varphi_n(t, 0) \geq (2-t) \frac{\partial \varphi_n}{\partial u}(t, 0) \geq \frac{\partial \varphi_n}{\partial u}(t, 0). \quad (2.13)$$

LEMMA 2.1 *If the law of η is infinitely divisible, then there exists $c > 0$ such that for every $\beta \in [0, \frac{B}{2})$,*

$$p'_n(\beta) \geq -\frac{c\beta}{n} \mathbb{E} E_{n,\beta}^{\otimes 2} L_n(S^1, S^2). \quad (2.14)$$

We postpone the proof of Lemma 2.1 to the next section. A simple calculation shows that

$$\frac{\partial \varphi_n}{\partial u}(t, 0) = \frac{\beta^2}{2n} \mathbb{E} E_{n,\sqrt{t}\beta}^{\otimes 2} L_n(S^1, S^2),$$

where $E_{n,\beta}$ is the expectation with respect to the probability measure $P_{n,\beta}$, and $E_{n,\beta}^{\otimes 2}$ is the expectation with respect to the product measure $P_{n,\beta}^{\otimes 2}$ on the product space $(\Sigma^2, \mathcal{E}^{\otimes 2})$. Since

$$\phi'_n(t) = \frac{\beta}{2\sqrt{t}} p'_n(\sqrt{t}\beta),$$

the lemma together with the inequality (2.13) imply that for every $t \in [0, 1]$,

$$\phi'_n(t) \geq -\frac{c\beta^2}{2n} \mathbb{E} E_{n,\sqrt{t}\beta}^{\otimes 2} L_n(S^1, S^2) \geq c(\varphi_n(t, 0) - \varphi_n(t, 2-t)), \quad (2.15)$$

provided we take $\beta < \frac{B}{2}$.

LEMMA 2.2 *If $\sigma^2 < 1$ then there exists $B_2 > 0$ such that for every fixed β in $(0, B_2)$,*

$$\frac{\partial}{\partial u} \varphi_n(t, u) - \frac{\partial}{\partial t} \varphi_n(t, u) \geq 0 \quad \text{for } t \in [0, 1] \text{ and } u \geq 0. \quad (2.16)$$

We postpone the proof of Lemma 2.2 and we conclude the proof of the theorem. We set $B_1 = \min(\frac{B}{2}, B_2)$ and fix β in $[0, B_1)$. Then according to Lemma 2.2

$$\varphi_n(t, 2-t) - \varphi_n(0, 2) = \int_0^t \frac{\partial \varphi_n}{\partial t}(s, 2-s) - \frac{\partial \varphi_n}{\partial u}(s, 2-s) ds \leq 0. \quad (2.17)$$

Combining (2.12), (2.15), and (2.17), we obtain (2.11), which ends the proof.

REMARK 2.3 If the law of η is standard normal, then the inequality (2.14) is an equality with $c = 1$, and the inequality (2.16) holds true. Both facts are shown in [10] by using gaussian integration by parts.

3 Proof of Lemma 2.1

According to [2], Proposition 11, there exists $c_2 > 0$ depending on β such that

$$p'_n(\beta) \geq -\frac{c_2}{n} \mathbb{E} E_{n,\beta}^{\otimes 2} L_n(S^1, S^2). \quad (3.18)$$

We recall rapidly the proof of [2]. We write

$$np'_n(\beta) = \mathbb{E} \frac{EH_n(S)e^{\beta H_n(S)}}{Ee^{\beta H_n(S)}} - n\lambda'(\beta) = \sum_{(i,x)} \mathbb{E} \eta(i, x) f_{i,x}(\eta(i, x)) - n\lambda'(\beta).$$

The distribution of η is infinitely divisible, therefore we have a Lévy Khinchine formula

$$\lambda(\beta) = c_0\beta + \frac{\sigma^2}{2}\beta^2 + \int (e^{\beta u} - 1 - \beta u \mathbf{1}_{|u| \leq 1}) \pi(du),$$

where $c_0 \in \mathbb{R}$, $\sigma \geq 0$, and π is a measure on \mathbb{R}^* such that $1 \wedge u^2 \in L^1(\pi)$. With these notations we have the following lemma.

Lemma ([2], Lemma 10). *For any bounded differentiable f with bounded derivative, one has the following integration by parts formula:*

$$\mathbb{E} \eta f(\eta) = c_0 \mathbb{E} f(\eta) + \sigma^2 \mathbb{E} f'(\eta) + \int_{\mathbb{R}} (\mathbb{E} f(\eta + u) - \mathbf{1}_{|u| \leq 1} \mathbb{E} f(\eta)) u \pi(du).$$

Applying the lemma, Carmona and al. calculate

$$np'_n(\beta) = -\sigma^2 \beta \mathbb{E} E_{n,\beta}^{\otimes 2} L_n(S^1, S^2) - \sum_{(i,x)} \mathbb{E} (P_{n,\beta}(S_i = x))^2 \int_{\mathbb{R}} \frac{(e^{\beta u} - 1) e^{\beta u} u}{(e^{\beta u} - 1) \mu_n(S_i = x) + 1} \pi(du).$$

Noting that

$$\sum_{(i,x)} (P_{n,\beta}(S_i = x))^2 = E_{n,\beta}^{\otimes 2} L_n(S^1, S^2),$$

and that

$$\begin{aligned} & \sup_{0 \leq a \leq 1} \int_{\mathbb{R}} \frac{(e^{\beta u} - 1) e^{\beta u} u}{(e^{\beta u} - 1) a + 1} \pi(du) \\ & \leq \int_{-\infty}^0 |u| (1 - e^{\beta u}) \pi(du) + \int_0^{+\infty} u e^{\beta u} (e^{\beta u} - 1) \pi(du) = c', \end{aligned} \quad (3.19)$$

which is finite provided that $\mathbb{E} e^{2\beta \eta}$ is finite, they conclude that (3.18) is true with $c_2 = c' + \sigma^2 \beta$.

Now for $u < 0$ we have $1 - e^{\beta u} \leq \beta |u|$, and for $u > 0$ we have $e^{\beta u} - 1 \leq \beta u e^{\beta u}$, so returning to (3.19) we deduce that for every $\beta \in [0, \frac{B}{2}]$,

$$c' \leq \beta \left(\int_{-\infty}^0 u^2 \pi(du) + \int_0^{+\infty} u^2 e^{B u} \pi(du) \right),$$

and conclude that (2.14) is satisfied with

$$c = \sigma^2 + \int_{-\infty}^0 u^2 \pi(du) + \int_0^{+\infty} u^2 e^{B u} \pi(du).$$

4 Proof of Lemma 2.2

We calculate

$$\frac{\partial}{\partial u}\varphi_n(t, u) - \frac{\partial}{\partial t}\varphi_n(t, u) = \frac{1}{2n}E^{\otimes 2}[e^{u\beta^2 L_n(S^1, S^2)}\mathbb{I}(S^1, S^2)], \quad (4.20)$$

where

$$\begin{aligned} \mathbb{I}(S^1, S^2) &= \mathbb{E} \frac{e^{\sqrt{t}\beta H_n(S^1, S^2) - 2n\lambda(\sqrt{t}\beta)}(\beta^2 L_n(S^1, S^2) - \frac{\beta}{2\sqrt{t}}H_n(S^1, S^2) + \frac{\beta n}{\sqrt{t}}\lambda'(\sqrt{t}\beta))}{E^{\otimes 2}e^{\sqrt{t}\beta H_n(S^1, S^2) - 2n\lambda(\sqrt{t}\beta) + u\beta^2 L_n(S^1, S^2)}} \\ &= \mathbb{E} \frac{e^{\sqrt{t}\beta H_n(S^1, S^2)}(\beta^2 L_n(S^1, S^2) - \frac{\beta}{2\sqrt{t}}H_n(S^1, S^2) + \frac{\beta n}{\sqrt{t}}\lambda'(\sqrt{t}\beta))}{E^{\otimes 2}e^{\sqrt{t}\beta H_n(S^1, S^2) + u\beta^2 L_n(S^1, S^2)}} \end{aligned}$$

We show that $\mathbb{I}(S^1, S^2)$ is non negative for every (S^1, S^2) in Σ^2 . Let (S^1, S^2) be fixed in Σ^2 . We define

$$I = \{i \in \llbracket 1, n \rrbracket; S_i^1 = S_i^2\} \text{ and } A = \{(i, x); 1 \leq i \leq n \text{ and } x = S_i^1 \text{ or } S_i^2\}.$$

We write H_n for $H_n(S^1, S^2)$ and L_n for $L_n(S^1, S^2)$. For $v = (v_{(i,x)})_{(i,x) \in A}$, we define

$$H_n(v) = \sum_{i=1}^n v_{(i, S_i^1)} + v_{(i, S_i^2)} = \sum_{i \in I} 2v_{(i, S_i^1)} + \sum_{i \notin I} v_{(i, S_i^1)} + v_{(i, S_i^2)}$$

Then the measure defined by

$$\mu(dv) = \frac{e^{\sqrt{t}\beta H_n(v)}}{\mathbb{E}(e^{\sqrt{t}\beta H_n})} \mathbb{P}_\eta^{\otimes (2n - L_n)}$$

is a product probability measure on $\mathbb{R}^{2n - L_n}$, and we can write $\mathbb{I}(S^1, S^2)$ as an expectation with respect to μ :

$$\begin{aligned} \mathbb{I}(S^1, S^2) &= \int \frac{\beta^2 L_n - \frac{\beta}{2\sqrt{t}}H_n(v) + \frac{\beta n}{\sqrt{t}}\lambda'(\sqrt{t}\beta)}{E^{\otimes 2}e^{\sqrt{t}\beta H_n(v) + u\beta^2 L_n}} \mathbb{E}(e^{\sqrt{t}\beta H_n}) \mu(dv) \\ &= \int X(v)Y(v) \mu(dv), \quad (4.21) \end{aligned}$$

with

$$X(v) = \frac{1}{E^{\otimes 2}e^{\sqrt{t}\beta H_n(v) + u\beta^2 L_n}}$$

and

$$Y(v) = \left(\beta^2 L_n - \frac{\beta}{2\sqrt{t}}H_n(v) + \frac{\beta n}{\sqrt{t}}\lambda'(\sqrt{t}\beta) \right) \mathbb{E}(e^{\sqrt{t}\beta H_n}).$$

As X and Y are decreasing random variables in $L^2(\mu)$, we deduce from the FKG inequality that

$$\int X(v)Y(v) \mu(dv) \geq \int X(v) \mu(dv) \times \int Y(v) \mu(dv). \quad (4.22)$$

We refer to [9], Theorem 2.4 or [11], chapter II for more information about the FKG inequality. We calculate $\int Y(v) \mu(dv)$ and find that

$$\int Y(v) \mu(dv) = \left(\beta^2 L_n + \frac{\beta n}{\sqrt{t}}\lambda'(\sqrt{t}\beta) \right) \mathbb{E}(e^{\sqrt{t}\beta H_n}) - \frac{\beta}{2\sqrt{t}} \mathbb{E}(H_n e^{\sqrt{t}\beta H_n}).$$

Using that $\mathbb{E}(\eta e^{u\eta}) = \lambda'(u)e^{\lambda(u)}$, we calculate

$$\mathbb{E}(H_n e^{\sqrt{t}\beta H_n}) = \left(2L_n \lambda'(2\sqrt{t}\beta) + 2(n - L_n) \lambda'(\sqrt{t}\beta)\right) \mathbb{E}(e^{\sqrt{t}\beta H_n}),$$

hence

$$\int Y(v) \mu(dv) = \beta^2 L_n \left(1 - \frac{\lambda'(2\sqrt{t}\beta) - \lambda'(\sqrt{t}\beta)}{\sqrt{t}\beta}\right) \mathbb{E} e^{\sqrt{t}\beta H_n}.$$

There exists $a \in (\sqrt{t}\beta, 2\sqrt{t}\beta)$ such that $\frac{\lambda'(2\sqrt{t}\beta) - \lambda'(\sqrt{t}\beta)}{\sqrt{t}\beta} = \lambda''(a)$. Since $\lambda''(0) = \sigma^2$ is strictly less than one there exists $B_2 > 0$ such that for every s in $[0, 2B_2]$, $\lambda''(s)$ remains strictly less than one. If $\beta \in (0, B_2)$ then $a \in (0, 2B_2)$, hence $\lambda''(a) < 1$, from which we deduce that $\int Y(v) \mu(dv)$ is non negative. As $\int X(v) \mu(dv)$ is also obviously non negative, recalling (4.20), (4.21) and (4.22), we get the result.

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